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ON THE DISTRIBUTION OF TITANIUM OXIDE AND CARBON STARS IN OUR PART OF THE MILKY WAY

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The Dearborn Observatory Survey of Faint Red Stars from four and a half degrees below the equator to the north pole has been completed. Some 44,000 stars of types K5, M0 to M9, R and N have been classified, and their magnitudes have been determined. Publication of the final half of these stars, and of a discussion of them all is under way.

In each of 317 areas magnitude limits ranging from 11 to nearly 13 were reached.

Upon plotting the K5 and M stars in galactic coördinates several items became apparent.

1. The obscuring clouds of interstellar material certainly affect the distribution decidedly. In fact, the axis of maximum frequency makes an angle of about 14 degrees with the galactic equator crossing it from south to north at longitude 90°. Other characteristics of distribution indicate the possibility that reasons, more basic than obscuration by cosmic clouds, may exist.

2. Among the 44,000 stars, 2894 belong in the advanced titanium oxide classes, M5 to M9. Of these, 1684 have apparent magnitudes brighter than 10 and only 1210 are of magnitude 10 or fainter. Dwarfs at distances from 100 to 1000 light years are too scarce to be statistically important in the observed frequencies. We are dealing mostly with giants, and beyond galactic latitudes of 30 degrees north and south they are nearly all in the brighter group. This survey has reached the galactic boundaries in these directions.

Along the broad galactic band we expected to find great numbers of distant red giants of these types recorded on our plates as faint stars. However, the ratio of numbers in the two magnitude groups is about 1, whereas the general stellation is 2 to 4 or more in favor of the fainter stars

Stars of types M5 to M9 are the easiest of all to detect and classify on our red sensitive plates. We probably missed very few in the magni-

tude range from 7 to 13. We noticed this preponderance of bright giants from the start and have mentioned it before. We are now, on the basis of all of our material, obtained systematically in the northern 54 per cent of the sky, forced to conclude that spectra showing the TiO band are particularly abundant in our part of the Milky Way and perhaps that man has appeared upon the cosmic scene in an era when many stars in his vicinity are going through their "titanium oxide stage."

In the course of this survey, most of the known carbon stars in our zones have been reobserved and 209 new ones have been catalogued. Of the latter we feel certain about 144–89 of type N and 55 of type R. The total number of carbon stars now known in the whole sky is about 418—282 of type N and 136 of type R. The galactic concentration of both kinds is very high.

As early as in 1941 we reported the discovery of a nest of N stars just east of Orion. Including all of these stars now known this cluster stands out very clearly. In an area of 750 square degrees, roughly pentagonal in shape, centering in galactic longitude 168°, latitude −5°, there are about 50 N stars, with only 2 R stars involved. On both sides of this nest along the galactic equator the two types are well mixed in frequency.

In Sanford's figure [*Astrophysical Journal*, **99**, 156 (1944)] this cluster is noticeable. The average magnitude of carbon stars in his list is about 9. Ours average around 11. Since R stars are about 2 magnitudes less luminous than the N stars a thin cloud of obscuring material in the foreground might conceivably blot out the former while allowing the latter stars to shine through. This does not explain the presence of the N star nest and would probably make it even more pronounced, actually, than we have observed it.

We seem to be driven to the conclusion that bodies containing an excess of carbon are relatively very abundant in the region of this nest of N stars.

THE SPREAD OF MEASLES IN THE FAMILY

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Some years ago, with the help of several collaborators, I discussed Major Greenwood's theory of the spread of measles within the family.[1] That theory had two postulates: (1) if a primary case (or several co-primary cases) brought measles into a family in which there were m other susceptibles, the chances that $m, m-1, \ldots, 0$ secondaries would result

would be p^m, $mp^{m-1}q$, ..., q^m as though the children were independently infected, the chance of infection being p; and (2) the secondary case or cases would expose the remaining susceptibles to infection for a subsequent generation of cases secondary to the secondaries (tertiary cases as we called them) with the same chance p. Greenwood had tested his theory against observation by comparing the observed and expected numbers of families which had 0, 1, 2, ... cases *altogether* beyond the primary and had found a good fit. On the same gross comparison we also found a good fit to his theory with our very different data, but when we broke down the data into successive generations as called for by the detailed hypotheses (1) and (2), we found that (1) the calculated distribution of the numbers of families by numbers of cases in each generation did not fit the observations; and (2) the tertiary attack rate was significantly lower than the immediate or direct secondary attack rate. So the theory became untenable for our Providence data. I wish to examine the matter a trifle further.

Tabulation of the samples II (p. 446), IV (p. 447), and VI (p. 448) of families containing two susceptibles according as the families have 0, 1, 2 immediate secondary cases gives:

CASES	0	1	2	TOTAL
For II	34	61	239	334
For IV	15	41	129	185
For VI	49	102	368	519

The χ^2-test shows that II and IV differ only about as much as might be expected by chance and may therefore be combined into VI without hesitation. The chance of a secondary case (derived from a primary) is $p = 0.807$ and the chance distribution of the 519 families would be 19, 162, 338 instead of 49, 102, 368. The hypothesis (1) that measles spreads in the family although the children were independently infected is therefore untenable.

One way to express dependence of elements is to compute the number which would be required by the theory of chance to explain the observed standard deviation. The actual secondary attack rates in the families with 0, 1, 2 secondary cases are 0, 0.5, 1.0, respectively; their mean is 0.807 and their standard deviation squared (variance) is 0.016. If this be equated to pq/n, we find $n = 1.46$. Thus the two susceptibles in the family are behaving relative to contracting or escaping the infection as though they were about one and one-half.[2]

Another, and presumably better, method is to make an analysis free from the hypothesis that the chances of incidence within the family are independent. Let the two susceptibles in each family be designated in some way as A and B (for example, A may be the older and B the younger). Then the families may be entered in a 4-fold table according as A and B or A but not B or B but not A or neither A nor B contracts the disease.

In the notation of Yule, using parentheses to denote the numbers of cases with specified attributes, we have the table:[3]

	A	α	
B	(AB)	(αB)	(B)
β	$(A\beta)$	$(\alpha\beta)$	(β)
	(A)	(α)	N

$p_A = (A)/N$, $q_A = (\alpha)/N$,
$p_B = (B)/N$, $q_B = (\beta)/N$.

By the theory of probabilities, when events are not independent, the chance of A and B is the chance of A multiplied by the chance of B if A occurs, or

$$p_{AB} = \frac{(AB)}{N} = \frac{(A)}{N} \cdot \frac{(AB)}{(A)}, \quad p_{\alpha\beta} = \frac{(\alpha\beta)}{N} = \frac{(\alpha)}{N} \cdot \frac{(\alpha\beta)}{(\alpha)}.$$

The probabilities p_{AB} and $p_{\alpha\beta}$ are observable as both or neither without specification of the susceptibles as A and B, but without such specification we cannot determine $(A\beta)$ and (αB) though we here can determine the chance p_1 of just one case and also the secondary attack rate s as

$$p_1 = \frac{(A\beta) + (\alpha B)}{N} \quad \text{and} \quad s = p_{AB} + {}^1/_2\, p_1.$$

If we will introduce $\eta = p_A - p_B$, the difference between the chances of A and of B, we may solve for all elements in terms of the observed quantities p_{AB}, $p_{\alpha\beta}$, p_1, s and the hypothetical quantity η as follows:

$$\text{Chance of } A \text{ is } p_A = s + {}^1/_2\,\eta, \text{ chance of } B \text{ is } p_B = s - {}^1/_2\,\eta,$$

$$\text{Chance of } B \text{ if } A \text{ is } \frac{p_{AB}}{s + {}^1/_2\,\eta}, \text{ chance of } \beta \text{ if } \alpha \text{ is } \frac{p_{\alpha\beta}}{s - {}^1/_2\,\eta}$$

It may be observed that, in a general association table, $\eta = p_A - p_B = p_{A\beta} - p_{\alpha B} = (A\beta)/N - (\alpha B)/N$, may range from $(A\beta)/N$ if (αB) happens to vanish to $-(\alpha B)/N$ if $(A\beta)$ happens to vanish.

In sample VI, there is no way to distinguish the two susceptibles and we may assume that they are in fact equivalent (as they probably would not be if distinguished by ages), that is $(A\beta)/N = (\alpha B)/N = 51/519 = 0.0982$ and $\eta = 0$. Then the secondary attack rate is 0.807, but in case one be assumed to be attacked the chance that the other will also be attacked is 0.88; and the escape rate is 0.193, but in case one be assumed to escape the chance that the other escape[4] is 0.49.

If we turn next to samples IX and XI (p. 450) with three susceptibles in addition to the primary, we have for immediate secondaries

CASES	0	1	2	3	TOTAL
For IX	4	11	18	67	100
For XI	4	16	8	27	55

The two samples differ enough so that it is very doubtful whether they may safely be combined into one (XIII). The secondary attack rate in IX is $s = 0.827$ and in XI is $s = 0.685$. If we compute the variances of the attack rates $p = 0, 1/3, 2/3, 1$ in the families, we find for IX and XI, respectively, 0.079 and 0.119; the respective values of pq are 0.143 and 0.216, and the value of the number of independent susceptibles turns out in either case to be $n = 1.8$ which is well below 3.

An analysis can be made in terms of probabilities. It may be supposed that the three susceptibles are distinguished as possessing attributes A, B, C. There are then the three probabilities[5]

$$p_A = \frac{(A)}{N}, \; p_{B \cdot A} = \frac{(AB)}{(A)}, \qquad p_{C \cdot AB} = \frac{(ABC)}{(AB)},$$

namely, the chance of A, the chance of B if A, the chance of C if A and B, and many others, where the number in a class possessing assigned attributes is divided by the number in a class in which one of those attributes is missing, for example $(A\beta\gamma)/(A\beta)$ which is "the chance of not-C if A and not-B be given." The general analysis would be needed in case the attributes (such as serial order in age) were assigned; for our data where there is no such assignment the only solution we shall seek is that for which the attributes are equivalent, i.e., $(A) = (B) = (C)$, etc.

Under the assumption of equivalence we find

For IX: $(ABC) = 67$, $(AB\gamma) = (A\beta C) = (\alpha BC) = 6$,
$(\alpha\beta\gamma) = 4$, $(A\beta\gamma) = (\alpha B\gamma) = (\alpha\beta C) = 11/3$;

For XI: $(ABC) = 27$, $(AB\gamma) = (A\beta C) = (\alpha BC) = 8/3$,
$(\alpha\beta\gamma) = 4$, $(A\beta\gamma) = (\alpha B\gamma) = (\alpha\beta C) = 16/3$.

For IX: $p_A = 0.827$, $p_{B \cdot A} = 0.88$, $p_{C \cdot AB} = 0.92$,
$p_\alpha = 0.173$, $p_{\beta \cdot \alpha} = 0.44$, $p_{\gamma \cdot \alpha\beta} = 0.52$;

For XI: $p_A = 0.68$, $p_{B \cdot A} = 0.79$, $p_{C \cdot AB} = 0.91$,
$p_\alpha = 0.32$, $p_{\beta \cdot \alpha} = 0.54$, $p_{\gamma \cdot \alpha\beta} = 0.43$.

It is seen that the chance of an additional direct secondary case increases with the number of cases already assumed among the susceptibles in either sample; but the chance of three escapes granted two is in sample XI less than that of two escapes granted one, albeit the numbers are so small that the escape rates other than p_α are poorly determined.

One could analyze in a similar manner the other samples in the monograph on measles in Providence; and if additional data were available for Providence or for other places so recorded as to make possible the tabulations that might be necessary, the analysis could be applied thereto. In

this way it would be possible to develop a detailed factual background on which some rational theory of the spread of measles in the family might in due time be built. Clearly the "family" might be a "schoolroom" or any other group of which a sufficient number of a specified type were available for study; but it may be long before sufficient numbers of groups other than families are available.

[1] Wilson, Edwin B., Bennett, Constance, Allen, Margaret, and Worcester, Jane, "Measles and Scarlet Fever in Providence, R. I., 1929–1934, with Respect to Age and Size of Family," *Proc. Amer. Philos. Soc., Philadelphia*, **80**, 357–476 (1939). See particularly pp. 441–453.

[2] Such a method has been used in economic statistics to discuss connectivity of fluctuations. See Wilson, Edwin B., "The Periodogram of American Business Activity," *Quarterly Journal of Economics*, May 1934, pp. 375–417, especially pp. 386, 409, 415.

[3] Yule, G. U., *Introduction to the Theory of Statistics*, or new edition by G. U. Yule and M. G. Kendall.

[4] In the symmetrical case where $(A\beta) = (\alpha B)$ and $p_A = p_B = s$, the condition $p_{AB}/s > /p_A$ that "the chance of B if A" exceeds "the chance of A" is equivalent to the statement that the 4-fold table be positively associated.

[5] One may use as notation subscripts—those preceding the dot designating the group of which the probability is specified, those following the dot designating the attributes considered as granted.

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*HYBRIDIZATION BETWEEN RANA PIPIENS FROM VERMONT AND EASTERN MEXICO**

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Cross fertilization experiments have been carried out among a number of geographic populations of *Rana pipiens* Schreber.[1] The intraspecific hybrids are normal if the distance between the localities from which the parents are obtained is relatively small. As the north-south, but not east-west, distance between the parents' places of origin is increased the hybrids exhibit progressively more severe defects in development. The most extreme cases previously observed involve crosses between Wisconsin (or Vermont) and Texas parents. If the cross is between the southern female and a northern male, the embryos are retarded in development and exhibit a considerable reduction of head structures. In the reciprocal cross, northern female and southern male, retardation in rate is observed, the circulatory system is defective, and head structures are greatly overdeveloped. The viability in both cases is low.

In order to ascertain if this gradient in hybrid inviability continues to

increase, crosses have been made between frogs from Mexico and Vermont. Material from two Mexican localities was available. One of these was Monterrey in Nuevo Leon, and the other was the Rio Axtla, near the village of Axtla, in San Luis Potosi. These localities are 450 and 750 miles, respectively, south of the locality in Texas from which my previous material was obtained. All of the experiments which will be reported were conducted at 19.1°C.

Experiment 1: Vermont ♀ × *Axtla* ♂.—The types of defects previously described[1] for hybrids derived from northern males and southern females were observed. The yolk plug was abnormally large, the neural plate was very long, the distance between the neural folds in the head region was wider than normal, and development was retarded in post-gastrula stages. Cytolysis began in the neural fold stage and by the time the controls were in Stage 14L,[2] 31 per cent had died. Most of the embryos did not develop beyond Stages 16–17. Those which were able to continue development had extremely large heads and suffered circulatory system defects. One abnormal embryo, from a total of 155, reached the feeding stage before it died.

Experiment 2: Vermont ♀ × *Axtla* ♂.—These hybrids were more normal than those of Experiment 1. There was considerable retardation in rate of development but most of the embryos reached Stage 20. These had gigantic heads. The mucous glands were very large and secreted a thick brown mucous which hung from them in curtains. This was interpreted as an indication of hyper-activity of the mucous glands. The mouth was frequently plugged with endodermal cells. Seventy-seven per cent survived for 10 days; 2 per cent survived for 21 days; none survived as long as 29 days.

Experiment 3: Vermont ♀ × *Axtla* ♂.—All of these hybrids exhibited some degree of cytolysis and a few died in Stages 14–16. Thirty-four per cent survived to 104 hours, a time when the controls were in Stage 20. These had greatly enlarged heads. Nine per cent had a beating heart but none had gill circulation. One embryo, from a total of 188, survived to the twelfth day. It died shortly thereafter.

Experiment 4: Vermont ♀ × *Axtla* ♂.—These embryos exhibited marked cytolysis as neurulae. All but two, from a total of 247, died as gastrulae or neurulae. One of these two died on the fifth day and the other on the twelfth day.

Experiment 5: Vermont ♀ × *Monterrey* ♂.—Some of the embryos cytolyzed as neurulae but most formed abnormal embryos with large heads and circulatory system defects. Mucous secretion was abundant. All of the embryos were dead by the eleventh day.

Experiment 6: Vermont ♀ × *Monterrey* ♂.—In the neural fold stages 65 per cent showed extensive cytolysis. Cellular debris was coming off

from the neural groove. At this time 40 per cent had a perforation in the ventral epidermis which exposed the underlying germ layers. Seventeen per cent survived to the seventh day but they were very abnormal. All were dead by the fifteenth day.

Experiment 7: Vermont ♀ × *Monterrey* ♂.—This experiment gave the best hybrids secured in crosses of Vermont and Mexican individuals. There was little cytolysis in early stages but the usual type of morphological defect as well as retardation in rate of development was observed in the later stages. Twenty-two per cent survived to the sixteenth day and ten per cent are still alive on the one hundred and thirty-first day. Some of these will probably transform.

Experiment 8: Vermont ♀ × *Monterrey* ♂.—The male used in this cross was the same as in Experiment 7. In contrast with the previous case, the embryos of this experiment were the most abnormal hybrids in the entire series of crosses. The majority formed exogastrulae and none developed beyond the gastrula stage.

Experiment 9: Axtla ♀ × *Vermont* ♂.—The typical defects previously described[1] in hybrids derived from southern females and northern males were observed. The yolk plug was very small, the neural plate was short, the head and head structures were greatly reduced, and development was retarded in post-gastrula stages. Hatching was delayed long beyond the morphological stage in which it would normally occur. Some of the hybrids apparently died because they were unable to escape from their jelly membranes. Retardation in rate of development to Stage 20 was estimated as 66 per cent (in three experiments involving Texas females and either Wisconsin or Vermont males the retardation was 11, 11 and 8 per cent). Thirteen per cent of the embryos survived to the fourteenth day. Most of these died shortly thereafter except for one (1 per cent) which is still alive on the one hundred and fifty-eighth day. It is an albino.

Experiment 10: Axtla ♀ × *Vermont* ♂.—The same female was used as in Experiment 9 and development was essentially the same in the two cases. Sixteen per cent survived to the fourteenth day. A single embryo (1 per cent) is still alive on the one hundred and fifty-eighth day. It is an albino.

The defects observed in the Axtla ♀ × Vermont ♂ hybrids are essentially the same as those observed in Texas ♀ × Vermont ♂ crosses. Although retardation of development is greater in the Axtla female hybrids than in the Texas female hybrids, the morphological defects seem somewhat more extensive in the latter.

The cross Vermont ♀ × Mexico ♂ has produced the most extreme abnormalities so far observed in *Rana pipiens* racial crosses. The frequency of exogastrulation and the beginning of cytolysis as early in development as the gastrula and neurula stages is noteworthy. This is the basis for the

conclusion that the gradient in intraspecific hybrid inviability increases in Mexican populations of *Rana pipiens*.

In these experiments with intraspecific hybrids, we have reached a point where the extreme geographically separated populations are behaving towards one another as two "good" species. The defects observed in the hybrids are so extensive that the possibility of gene exchange between the two parent populations is most unlikely. At the same time adjacent populations appear to be perfectly interfertile. This paradox has been discussed previously and the conclusion reached that natural selection is promoting the development of different temperature-adapted races. The defects observed in the hybrids between northern and southern individuals are interpreted as a result of incompatibility in the interaction of a "low temperature genome" and a "high temperature genome" in the same zygote. The fact that the peripheral members are reproductively isolated from one another is merely a feature incidental to their adaptation.

It seems unlikely that isolating mechanisms such as sexual isolation, hybrid inviability and hybrid sterility are ever the *initial* isolating mechanisms in dividing a panmictic population of animals which reproduce sexually. Initial reproductive isolating mechanisms could best arise when the different parts of the population are physically separated through geographical, habitat or seasonal isolation. Under these conditions natural selection would be increasing the frequency of the genetic changes, which will subsequently act as isolating mechanisms, not for any value as isolating mechanisms but because they have some adaptive value in their special environment. A subsequent breakdown in effectiveness of geographical, habitat or seasonal isolation would bring two differently adapted populations into competition. If these produce hybrids which are inferior to the parental forms, then natural selection would favor the perfection of any differences which could serve as isolating mechanisms. In this manner the two "adaptive peaks" would remain distinct, and in fact be two species.

* Aided by a grant from the Penrose Fund of the American Philosophical Society.

[1] Moore, J. A., *Genetics*, **31**, 304–326 (1946).

[2] These stages of development are described in Pollister, A. W., and Moore, J. A., *Anat. Rec.*, **68**, 489–496 (1937).

A UNIQUENESS THEOREM FOR EIGENFUNCTION EXPANSIONS

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Let $w_n(x, y)$ be the complete normal orthogonal eigenfunctions of the boundary value problem

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \mu w = 0 \tag{1}$$

$$w(x, y) = 0 \text{ on } C. \tag{2}$$

Where C is the boundary of a regular region R, the eigenvalues μ_n being arranged in non-decreasing order of magnitude. If $f(x, y)$ be an arbitrary summable function defined in R, we may write

$$f(x, y) \sim \sum a_n w_n(x, y) \tag{3}$$

$$a_n = \int_D \int f w_n dx dy \tag{4}$$

the series on the right of (3) being called the *Fourier Eigenfunction Series* and a_n the *Fourier Coefficients* of $f(x, y)$. I have studied elsewhere[1] the problem of convergence and summability of a Fourier Eigenfunction Series. In this note I am interested in announcing a result on uniqueness of eigenfunction expansion. Actually, we have the following,

THEOREM. *Let us suppose we are given an eigenfunction series*

$$a_1 w_1(x, y) + a_2 w_2(x, y) + a_3 w_3(x, y) + \ldots, a_1, a_2, \ldots \text{ real} \tag{5}$$

satisfying the following properties

(*i*) *There exists a continuous function $\phi(x, y)$ defined on $D + C$ and vanishing on C such that*

$$\phi(x, y) \sim \sum \frac{a_n}{\mu_n} w_n(x, y) \tag{6}$$

(*ii*) *At every point in the region the series (5) is summable $(J_1^2, \lambda_n)^2$ to a bounded measurable function $f(x, y)$, i.e.,*

$$\lim_{t \to 0} 4 \sum a_n w_n \frac{J_1^2(\lambda_n t)}{(\lambda_n t)^2} = f(x, y), \qquad \lambda_n = (\mu_n)^{1/2} \tag{7}$$

the series on the left converging for $t > 0$.

Then (5) is the Fourier eigenfunction series of $f(x, y)$.

We indicate the general outlines of the proof:

The sum of the convergent series on the left of (7) defines, for every $t > 0$, a linear transformation of ϕ:

$$4\sum a_n w_n \frac{J_1^2(\lambda_n t)}{(\lambda_n t)^2} = \mathsf{U}_t(\phi) \qquad \text{say} \tag{8}$$

Then we have the following lemmas:

LEMMA 1. *If for every (x, y) in D, $\overline{\lim_{t\to 0}}\ \mathsf{U}_t(\phi) \geq 0$ then ϕ is subharmonic. Similarly if $\underline{\lim_{t\to 0}}\ \mathsf{U}_t(\phi) \leq 0$ then ϕ is superharmonic.*

So if $\mathsf{U}_t(\phi) \to 0$ as $t \to 0$ then ϕ is harmonic.

LEMMA 2. *If $\overline{\lim}\ \mathsf{U}_t[\phi] \geq c$ for every (x, y) in R implies $\mathsf{U}_t(\phi) \geq c$ for some region contained in R with a similar conclusion if $\underline{\lim}\ \mathsf{U}_t(\phi) \leq c$.*

From these lemmas we deduce that if $|f(x, y)| \leq M$, hypotheses (i) and (ii) of the theorem imply

$$|\mathsf{U}_t(\phi)| \leq M.$$

Therefore

$$4a_n \frac{J_1^2(\lambda_n t)}{(\lambda_n t)^2} = \int\int_R \mathsf{U}_t(\phi) w_n dx dy$$

and letting $t \to 0$ we obtain the theorem.

We might add in conclusion, hypothesis (i) will be fulfilled if a_n tends to zero as rapidly as $\mu_n^{-\epsilon}$ or $(\log \mu_n)^{-1-\epsilon}$ or $(\log \mu_n)^{-1}(\log\log \mu_n)^{-1-\epsilon}$, etc. Thus if a_n satisfies any one of these conditions and $\sum a_n w_n(x, y)$ converges everywhere to zero then $a_n = 0$; $n = 1, 2, \ldots$.

In the case of a double trigonometric series $\sum_{-\infty \ldots -\infty}^{+\infty \ldots +\infty} C_{\mu,\nu} \exp i(\mu x + \nu y)$ $C_{00} = 0$ the theorem takes the following form:

(i) $\sum \frac{C_{\mu\nu}}{\mu^2 + \nu^2} \exp i(\mu x + \nu y)$ is the *F.S.* of a continuous periodic function $\phi(x, y)$.

(ii) $\lim_{t\to 0} 4 \sum_1^\infty \frac{J_1^2(t\sqrt{n})}{nt^2} \sum_{\mu^2+\nu^2=n} \frac{C_{\mu\nu}}{\mu^2 + \nu^2} \exp i(\mu x + \nu y) = f(x, y)$

at every point (x, y) where $f(x, y)$, is a bounded measurable function. Then the trigonometric series is the *F.S.* of $f(x, y)$.

[1] Cf. "Notes on Fourier Expansions (1)," *Jour. Lond. Math. Soc.*, **20**, 148–153 (1945) and the references therein.

[2] For definition of Bessel Summability, cf. the author's paper, "A New Summation Process," *Math. Student.*, **2**, 21–27 (1943).

ON THE CHARACTERISTIC CLASSES OF RIEMANNIAN MANIFOLDS

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The following imbedding theorem of sphere bundles was first stated, together with a sketched proof, by H. Whitney:[1] To a given bundle of spheres of dimension $n - 1$ over a compact manifold M there exists a mapping $f(M) \subset H(n, N)$ (which is the Grassmann manifold of all oriented linear spaces of dimension n through the origin of a real oriented Euclidean space E^{n+N} of $n + N$ dimensions), which defines a sphere bundle over M equivalent to the given one, provided that dim $M \leqq N$. Steenrod[2] proved that if dim $M \leqq N - 1$, then two sphere bundles over M defined by the mappings $f_i(M) \subset H(n, N)$, $i = 1, 2$, are equivalent, when and only when the mappings f_1 and f_2 are homotopic. From these theorems it would follow immediately that the inverse homomorphism f^* of the cohomology ring of $H(n, N)$ into the cohomology ring of M is independent of the mapping f in the process of imbedding. But it was Pontrjagin[3] who first explicitly made this observation for the case of tangent sphere bundles and defined to be a characteristic cohomology class of M the image of a cohomology class of $H(n, N)$ under the inverse homomorphism f^*. These characteristic cohomology classes include the ones which were first studied by Stiefel[4] and Whitney,[5] but they give many new ones.

Pontrjagin was interested mainly in tangent sphere bundles. He proved the imbedding theorem of tangent sphere bundles by simply imbedding the manifold M into E^{n+N} according to Whitney's imbedding theorem of differentiable manifolds. By being so imbedded, M acquires an induced Riemannian metric. In a second note[6] Pontrjagin showed that certain characteristic classes of M are expressible, in the sense of de Rham, by exterior differential forms constructed from the induced Riemannian metric. It is to be observed here that these characteristic classes are the images under f^* of the classes of $H(n, N)$ which form a basis of the cohomology ring of $H(n, N)$ with rational coefficients. This result of Pontrjagin includes, for instance, the now well-known generalization of the formula of Gauss-Bonnet for the case of an imbedded Riemannian manifold, as was first proved by Allendoerfer[7] and Fenchel.[8]

It is, however, not known whether the same results between the characteristic classes of M and the exterior differential forms of the Riemannian metric of M remain true, if the Riemannian metric is defined intrinsically on M. This question is significant, because the Riemannian metric of a differentiable manifold is in general not defined by imbedding the manifold

in a Euclidean space and it is not known whether an intrinsic Riemannian metric can be defined as an induced metric (in a Euclidean space), when the whole manifold is under consideration. It is the aim of this note to sketch a proof of the theorem that Pontrjagin's results remain true when the Riemannian metric on M is an intrinsic one. Complete details, together with generalizations to non-compact manifolds and affinely connected manifolds, will be published elsewhere.

We begin by stating the results of Pontrjagin in question. Let $\Omega_{ik}(i, k = 1, \ldots, n)$ be the curvature forms of the Riemannian metric of M, in the notation of Elie Cartan. By exterior multiplication we construct from them the differential forms

$$\Delta_{4m} = (1/c_{4m}) \sum \Omega_{i_1 i_2} \Omega_{i_2 i_3} \ldots \Omega_{i_{2m} i_1}, \qquad 1 \leq m \leq n/4,$$

$$\Delta_0 = \sum \epsilon_{i_1 \ldots i_n} \Omega_{i_1 i_2} \ldots \Omega_{i_{n-1} i_n}, \qquad \text{if } n \text{ is even},$$

where $\epsilon_{i_1 \ldots i_n}$ is the Kronecker index and where the summations are extended over all the indices i from 1 to n. The number c_{4m} is a constant which is so chosen as to make the cocycle Δ_{4m} an integral cocycle. In fact, c_{4m} is equal to $m!$ times the total measure of all the straight lines in a spherical space of $m + 1$ dimensions. The theorem, which Pontrjagin proved for the case of an induced Riemannian metric and which we shall prove to be true in general, is the following:

THEOREM. *Let M be a compact orientable Riemannian manifold. The differential forms Δ_0, Δ_{4m} $(1 \leq m \leq n/4)$ define cohomology classes which are characteristic cohomology classes of M.*

To prove this theorem we shall first give a description of the cohomology classes of $H(n, N)$. We take a sequence of integers $a_1, \ldots, a_n$ such that

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$$

and a sequence of linear spaces

$$L_1 \subset L_2 \subset \ldots \subset L_n$$

through the origin 0 of E^{n+N}, whose dimensions are

$$\dim L_i = N + i - a_i, \qquad i = 1, \ldots, n.$$

In $H(n, N)$ we consider the linear spaces X such that

$$\dim (X \cap L_i) \geq i, \qquad 1 \leq i \leq n.$$

The totality of these linear spaces, minus the ones on the boundary, forms a cell of dimension $nN - \sum_{1=i}^{n} a_i$. In particular, when the a's satisfy certain conditions[9] which amount to say that certain integers constructed from them are even, the cell defines a cycle of dimension $nN - \sum_{i=1}^{n} a_i$ and hence a

cocycle of dimension $\sum_{i=1}^{n} a_i$ on $H(n, N)$, since $H(n, N)$ is orientable. The corresponding cohomology class we shall denote by the symbol $[a_1 \ldots a_n]$. Of particular importance for our purpose will be the cohomology classes $\Gamma_{4m} = [2 \ldots 2\ 0 \ldots 0]$ of dimension $4m$, where the symbol contains $2m$ 2's followed by $n - 2m$ 0's, $m = 1, \ldots, \left[\frac{n}{4}\right]$.

With these preliminaries we shall give the main steps of the proof in a number of lemmas.

LEMMA 1. *Over the base manifold M let $\mathfrak{F}$ be the fibre space of the ordered sets of $n - 2m + 2$ vectors satisfying the condition that they do not belong to a linear subspace of dimension $n-2m$. If F denotes a fibre of $\mathfrak{F}$, then the i-dimensional homotopy groups $\pi_i(F)$ of F, $0 < i \leqq 4m-2$, vanish, while the $(4m-1)$-dimensional homotopy group $\pi_{4m-1}(F)$ is free cyclic. The cohomology class Γ_{4m} is the "obstacle cohomology class" (called by Steenrod[10] the characteristic class) of $\mathfrak{F}$ in M.*

In fact, by a proper extension of the covering homotopy theorem to fibre spaces with exceptional fibres, it is not hard to show that F is simply connected. By a theorem of Hurewicz[11] the statements about the other homotopy groups reduce to corresponding statements about homology groups of the same dimension. By a retraction we can suppose that the sum of squares of all the components of the $n - 2m + 2$ vectors is equal to one, so that F becomes a subspace of a sphere of $n(n - 2m + 2) - 1$ dimensions. It then follows from Alexander's duality theorem that the homology groups $H^i(F) = 0$, $0 < i \leqq 4m - 2$, and that $H^{4m-1}(F)$ is free cyclic. A generator of $H^{4m-1}(F)$ can be given, by the explicit calculation of a linking coefficient. The last statement of the lemma follows almost immediately from a comparison of the definitions of the two cohomology classes.

LEMMA 2. *In the fibre bundle of all rectangular frames over M the differential form Δ_{4m} is equal to the exterior derivative of a differential form of degree $4m - 1$, which reduces on a fibre to the form*

$$\frac{1}{e_{4m}} \sum \omega_{i_1 i_2} \ldots \omega_{i_{4m-1} i_1},$$

where $e_{4m} \neq 0$ is a constant and where ω_{ij} are the infinitesimal components of the frames.

The proof of this lemma is purely algebraic. It follows from certain identities for the exterior derivatives of some combinations of the differential forms. A particularly important rôle is played by these combinations which are formed from w_{ij}, Ω_{ij}, by a "cyclic summation" of the indices.

LEMMA 3. *In the group manifold $O(n)$ of the proper orthogonal group in n*

variables let $O(n - 2m + 2)$ be a subgroup imbedded in $O(n)$ in a definite way (for instance, by keeping the last $2m - 2$ variables invariant). If $4m \leqq n$, every cycle of dimension $4m - 1$ of $O(n)$ is homologous to a cycle of $O(n - 2m + 2)$.

This lemma follows from well-known results of Pontrjagin.[12]

Our theorem will be proved if we can identify Γ_{4m} with Δ_{4m}. For this purpose we take a differentiable simplicial cycle z of dimension $4m$ and show that its Kronecker product with Γ_{4m} is equal to the integral of Δ_{4m} over it. z can be supposed to be in such a general position that there is a continuous field of $n - 2m + 2$ vectors defined over z with only isolated points (to be called singular points) at which these vectors belong to a linear space of dimension $n - 2m$. The locus in M at which the $n - 2m + 2$ vectors are linearly dependent is in general of dimension $n - 2m + 1$ and its intersection with z of dimension $2m + 1$. It is therefore possible to suppose the vector field so chosen that the points at which the $n - 2m + 2$ vectors are linearly dependent belong to the $(4m - 1)$-dimensional skeleton of z. Under these assumptions which do not affect the proof, we can deform the vector field into a field such that the $n - 2m + 2$ vectors are mutually perpendicular unit vectors at each interior point of a $4m$-dimensional simplex of z, with the possible exception of one singular point.

In each (open) simplex of z there is at the same time defined a continuous field of $(2m - 2)$-dimensional planes with the possible exception of one singular point, namely the field of planes perpendicular to the $n - 2m + 2$ vectors defined above. It is then possible to define a continuous field of ordered sets of $2m - 2$ mutually perpendicular unit vectors which lie in these planes and which are such that they give rise to no new singularity.

To each simplex σ_i of z we construct a differentiable function (a "density") λ_i, $0 \leqq \lambda_i \leqq 1$, such that $\lambda_i = 0$ on the boundary of σ_i and that the content of the points of σ_i at which $\lambda_i \neq 1$ approaches zero, where content is understood in the sense of Whitney. We define $\lambda_\tau = 0$ for all points $p \in z$, which do not belong to σ_i. Then the integral $\int_z \Delta_{4m}$ will differ as small as we please from the integral

$$\int_z (\sum_i \lambda_i) \Delta_{4m} = \sum_i \int_{\sigma_i} \lambda_i \Delta_{4m}.$$

Now in σ_i we have defined a continuous field of rectangular frames with a possible singularity at an interior point. We consider over σ_i the fibre bundle F^* of frames and integrate $\lambda_i \Delta_{4m}$ in F^*. From Lemma 2 it is seen that $\lambda_i \Delta_{4m}$ can be written as the exterior derivative of a differential form of degree $4m - 1$. The value of this integral is therefore contributed by the singularity and it follows from Lemma 3 that it is exactly the value assigned to σ_i by the cocycle Γ_{4m}. In this way we prove that $\int_z \Delta_{4m}$ is equal to the Kronecker product of Γ_{4m} with z.

In conclusion we observe that it is not essential to assume our manifold to be compact, provided that the integration is carried out only over finite cycles. We also observe that similar results hold for affinely connected manifolds.[13]

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ON THE FUNCTIONAL EQUATION $\frac{\partial}{\partial z}F(z, \alpha) = F(z, \alpha + 1)$

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We shall study differential-difference equations of the type

$$\frac{\partial}{\partial x} f(x, \alpha) = A(x, \alpha)f(x, \alpha) + B(x, \alpha)f(x, \alpha + 1), \qquad (1)$$

where $A(x, \alpha)$ and $B(x, \alpha)$ are given functions. Included in equations of type (1) are the differential recurrence relations satisfied by Bessel, Legendre, Laguerre and Hermite functions. We shall motivate, discover and coördinate many of the formal properties of these functions by showing that they are special cases of a few simple formulae satisfied by certain classes of solutions of equations of the type (1). In this note we state the principal results only; proofs, additional theorems and examples, and explanations will appear in "An Essay on the Functional Equation $\frac{\partial}{\partial z} F(z, \alpha) = F(z, \alpha + 1)$," forthcoming in the *Annals of Mathematics Studies*.

If $f(x, \alpha)$ is a solution of the equation (1), and if

$$\mathbf{f}(x, \alpha) \equiv \exp\{-\textstyle\int_{x_0}^{x} A(v, \alpha)dv\} f(x, \alpha) \qquad (2)$$

then $\mathbf{f}(x, \alpha)$ satisfies the equation

$$\frac{\partial}{\partial x}\mathbf{f}(x, \alpha) = B(x, \alpha) \exp\{\int_{x_0}^{x} \Delta_{\alpha} A(v, \alpha)dv\}\mathbf{f}(x, \alpha + 1). \quad (3)$$

Hence we lose no generality in studying equations of the type

$$\frac{\partial}{\partial x}\mathbf{f}(x, \alpha) = C(x, \alpha)\mathbf{f}(x, \alpha + 1), \quad (4)$$

where $C(x, \alpha)$ is a given function. We consider only the case when $C(x, \alpha)$ is factorable:

$$C(x, \alpha) = X(x)A(\alpha). \quad (5)$$

If we make the change of variables

$$z \equiv \int_{x^1}^{x} X(v)dv, \quad (6)$$

$$F(z, \alpha) \equiv \exp\{\sum_{\alpha_0}^{\alpha} \log A(v)\,\Delta v\}\mathbf{f}(x, \alpha) \quad (7)$$

then we find that

$$\frac{\partial}{\partial z}F(z, \alpha) = F(z, \alpha + 1). \quad (8)$$

The equation (8), which we shall call "the F-equation," is the subject of our study.

THEOREM 1. *The factorability condition (5) is equivalent to the following condition on the coefficients of the original equation (1):*

$$\frac{\partial}{\partial x} \log B(x, \alpha) + \Delta_{\alpha} A(x, \alpha) = \chi(x), \quad (9)$$

where $\chi(x)$ is a function of x only.

The condition (9) is fulfilled by the coefficients of nearly all the differential-difference equations of type (1) satisfied by familiar special functions.

THEOREM 2. *Suppose $f(x, \alpha)$ satisfies an equation of the type (1). If by transformations of the type*

$$F(z, \alpha) \equiv M(x, \alpha)f(z, \alpha), \qquad z \equiv \phi(x), \quad (10)$$

there may be derived from $f(x, \alpha)$ two different functions $F_1(z, \alpha)$ and $F_2(z, \alpha)$ both satisfying the F-equation, then there exist constants k and l and a periodic function $\pi(\alpha)$ of period 1 such that

$$F_2(z, \alpha) = \pi(\alpha)k^{\alpha}F_1(kz + l, \alpha). \quad (11)$$

Solutions of the F-Equation Involving Familiar Functions.—In the follow-

ing list z and α are the (complex) variables of the F-equation, and any other parameters γ, δ, etc., are constants

	EXPRESSION	NAME OF FUNCTION INVOLVED
1.	$e^{i\alpha\pi} z^{-\alpha/2}\, \mathbf{Z}_\alpha(2\sqrt{z})$	Bessel or Hankel function of either kind
2.	$z^{-\alpha/2}\, \mathbf{Z}_{-\alpha}(2\sqrt{z})$	
3.	$e^{i\alpha\pi} e^{-z}\, L_\gamma^{(\alpha)}(z)$	Laguerre function
4.	$z^{-\alpha}\, L_\gamma^{(-\alpha)}(z)/\Gamma(\gamma - \alpha + 1)$	
5.	$\Gamma(\alpha + 1)(-z)^{-\alpha-1-\gamma}\, e^{-1/z}\, L_\alpha^{(\gamma)}\left(\frac{1}{z}\right)$	
6.	$\Gamma(\alpha - \gamma)(-z)^{-\alpha}\, L_{-\alpha}^{(\gamma)}\left(\frac{1}{z}\right)$	
7.	$e^{-2\sqrt{z}}\, {}_1F_1(\alpha; 2\alpha; 4\sqrt{z})/\Gamma(\alpha + 1/2)$	Confluent hypergeometric function
8.	$e^{-z^2} H_\alpha(-z)$	Hermite function
9.	$\Gamma(\alpha) H_{-\alpha}(-z/2)$	
10.	$\Gamma(\alpha - \beta + 1)(z^2 - 1)^{-\alpha+1/2}\, \mathfrak{P}_\alpha^\beta\left(-\frac{z}{\sqrt{z^2-1}}\right)$	Associated Legendre function of either kind
11.	$\Gamma(\alpha - \beta)(z^2 - 1)^{-\alpha/2}\, \mathfrak{P}_{-\alpha}^\beta\left(-\frac{z}{\sqrt{z^2-1}}\right)$	
12.	$(1 - z^2)^{-\alpha/2}\, \mathfrak{P}_\beta^\alpha(z)$	

There are at least 26 more solutions involving familiar special functions, some of these being functions occurring in the theory of numbers and in mathematical statistics.

We now list a number of theorems concerning solutions of the F-equation. In the examples following the theorems the numbers in parentheses refer to the above list of solutions, and indicate that the solution(s) of that(those) number(s) need only be substituted for $F(z, \alpha)$ in the formula given by the theorem in order for the formula given as an example to follow immediately. Of these examples, formulae (15), (27), (33), (42), (43), (48), (49), (55) and (77) are to the best of my knowledge new results, while formulae (16), (32), (37), (38), (39), (52), (57), (58), (60), (61), (67), (69), (70) and (72) are generalizations of known results.

It is very easy to prove any one of these formulae, once one sees it set up. In the study of special functions the difficulty lies first in discovery and second in coördination of results. In no case in applying the theorems of this note do we need to know in advance the formula that will result in a particular example. We ask simply for a formula of a general type, e.g., an infinite exponential integral involving Laguerre functions, a contour integral for Bessel functions or a formula expressing Laguerre functions in terms of Bessel functions, and each time we find the resulting expression in a nearly automatic fashion. The purpose of all examples is to illustrate the

discovering and correlative value of the theorems. There are many more interesting examples than those given here.

Latin m, n, p, r represent positive integers only; Greek letters always represent complex quantities.

THEOREM 3. *If $F(z, \alpha)$ satisfies the F-equation, then*

$$F(z, \alpha + n) = \frac{\partial^n}{\partial z^n} F(z, \alpha). \tag{12}$$

Examples 3a: (1, 2; Bessel)

$$(-)^n x^{-\alpha-n} J_{\alpha+n}(x) = \frac{d^n}{(xdx)^n} [x^{-\alpha} J_\alpha(x)]. \tag{13}$$

$$x^{\alpha-n} J_{\alpha-n}(x) = \frac{d^n}{(xdx)^n} [x^\alpha J_\alpha(x)]. \tag{14}$$

Example 3b: (5)

$$L_n^{(\gamma)}\left(\frac{1}{x}\right) = \frac{(-)^n}{n!} x^{1+\gamma} e^{1/x} \frac{d^n}{dx^n} [x^{-1-\gamma} e^{-1/x}]. \tag{15}$$

Examples 3c: (10; known special case of formula (16): $\alpha = 0$)

$$P_{\alpha+n}^{\alpha} (\cos \theta) = \frac{(-)^n \Gamma(2\alpha + 1)}{n! 2^\alpha \Gamma(\alpha + 1)} \csc^{\alpha+n+1} \theta \frac{d^n \sin^{2\alpha+1}\theta}{d(\cot \theta)^n} \tag{16}$$

$$Q_n(\cos \theta) = \frac{(-)^n}{n!} \csc^{n+1}\theta \frac{d^n}{d(\cot \theta)^n} \{\sin \theta \log (\cot \theta + \csc \theta)\}. \tag{17}$$

THEOREM 4. *If $\phi(\alpha)$ be such that*

$$\overline{\lim_{n \to \infty}} \frac{\phi(\alpha_0 + n + 1)}{n\phi(\alpha_0 + n)} = \frac{1}{k} \tag{18}$$

then whenever $|z - z_0| < k$ and $\alpha = \alpha_0 + i$, $i = 0, 1, 2, \ldots$, there exists a unique solution $F(z, \alpha)$ of the F-equation such that $F(z_0, \alpha) = \phi(\alpha)$. This solution may be represented by the power series

$$F(z, \alpha) = \sum_{n=0}^{\infty} \phi(\alpha + n) \frac{(z - z_0)^n}{n!}. \tag{19}$$

Examples 4a: (3, 7; Kummer)

$$e^z {}_1F_1(-\gamma; \alpha; z) = F(\alpha + \gamma; \alpha; -z). \tag{20}$$

$$e^{-2\sqrt{z}} {}_1F_1(\alpha; 2\alpha; 4\sqrt{z}) = {}_0F_1(\alpha + 1/2; z). \tag{21}$$

THEOREM 5 (Doetsch). *In Theorem 4, $F(z, \alpha)$ may be represented in the form*

$$F(z,\ \alpha)\ =\ e^{z-z_0}\ \sum_{n=0}^{\infty}\ \frac{(z\ -\ z_0)^n}{n!}\ \Delta^n\phi(\alpha). \tag{22}$$

COROLLARY (Euler). *If the series on the left is convergent,*

$$e^{-z}\ \sum_{n=0}^{\infty}\ a_n\ \frac{x^n}{n!}\ =\ \sum_{n=0}^{\infty}\ \Delta^n a_0\ \frac{x^n}{n!}. \tag{23}$$

THEOREM 6. *If $\phi(\alpha)$ is representable in a Newton series,*

$$\phi(\alpha)\ =\ \sum_{n=0}^{\infty}\ (-)^n a_n \binom{\alpha\ -\ 1}{n}, \qquad \Re\alpha\ \geqq\ \alpha_0, \tag{24}$$

then in Theorem 4 $F(z,\ \alpha)$ may also be represented by a Newton series:

$$F(z,\ \alpha)\ =\ \sum_{n=0}^{\infty}\ (-)^n a_n(z\ -\ z_0) \binom{\alpha\ -\ 1}{n}, \qquad \Re\alpha\ \geqq\ \alpha_0, \tag{25}$$

where

$$a_n(t)\ \equiv\ \sum_{p=0}^{\infty}\ (-)^p a_{n+p}\ \frac{t^p}{p!}. \tag{26}$$

Example 6: (For $\phi(\alpha)$ put $F(1\ -\ \alpha,\ \beta;\ \gamma;\ \delta)$, and rearrange.)

$$L_p^{(m+\epsilon)}(z)\ =\ (-)^m\ \sum_{n=0}^{m}\ (-)^n \binom{m}{n} L_{m+p}^{(n+\epsilon)}(z). \tag{27}$$

THEOREM 7. *If in Theorem 4 $\phi(\alpha)$ may be represented as a contour integral,*

$$\phi(\alpha)\ =\ \int_C \psi_t(\alpha)dt, \qquad \Re\alpha\ \geqq\ \alpha_0, \tag{28}$$

and if $|\psi_t(\alpha)|\ <\ M$ when $\Re\alpha\ \geqq\ \alpha_0$, then

$$F(z,\ \alpha)\ =\ \int_C\ \sum_{n=0}^{\infty}\ \psi_t(\alpha\ +\ n)\ \frac{(z\ -\ z_0)^n}{n!}\ dt, \qquad \Re\alpha\ \geqq\ \alpha_0. \tag{29}$$

Example 7a: (1; Schläfli)

$$J_\alpha(2\sqrt{z})\ =\ \frac{z^{\alpha/2}}{2\pi i}\ \int_{-\infty}^{(0+)}\ t^{-\alpha-1}\exp\left(t\ -\ \frac{z}{t}\right)dt. \tag{30}$$

Example 7b: (3; Sonine)

$$L_\beta^{(\alpha)}(z)\ =\ \frac{\Gamma(\alpha\ +\ \beta\ +\ 1)}{\Gamma(\beta\ +\ 1)2\pi i}\ \int_{-\infty}^{(0+)}\ \left(1\ -\ \frac{z}{t}\right)^\beta\ t^{-\alpha-1}\ e^t\ dt. \tag{31}$$

Example 7c: (1; case $\beta\ =\ \alpha\ +\ {}^1/_2$: Hankel)

$$J_\alpha(2\sqrt{z}) = \frac{-e^{-i\alpha\pi}z^{\beta/2}}{4i\Gamma(\beta)\sin\beta\pi\sin(\alpha-\beta)\pi} \int_A^{(1+,\,0+,\,1-,\,0-)} t^{(\alpha-\beta)/2}(1-t)^{\beta-1}J_{\alpha-\beta}(2\sqrt{zt})dt. \quad (32)$$

Example 7d: (3)

$$L_\beta^{(\alpha)}(z) = \frac{e^{-i\alpha\pi}\Gamma(\alpha+\beta+1)}{4i\Gamma(\gamma)\Gamma(\alpha+\beta+1-\gamma)\sin\gamma\pi\sin(\alpha-\gamma)\pi} \int_A^{(1+,\,0+,\,1-,\,0-)} t^{\alpha-\gamma}(1-t)^{\gamma-1}L_\beta^{(\alpha-\gamma)}(zt)dt. \quad (33)$$

THEOREM 8. *If $F(z, \alpha)$ is an analytic solution of the F-equation, then*

$$F(z+h, \alpha) = \sum_{n=0}^{\infty} \frac{h^n}{n!} F(z, \alpha+n). \quad (34)$$

Examples 8a: (1, 2; Lommel)

$$(z+h)^{-\alpha/2}J_\alpha(2\sqrt{z+h}) = \sum_{n=0}^{\infty} \frac{(-h)^n}{n!} z^{-(\alpha+n)/2} J_{\alpha+n}(2\sqrt{z}). \quad (35)$$

$$(z+h)^{\alpha/2}J_\alpha(2\sqrt{z+h}) = \sum_{n=0}^{\infty} \frac{h^n}{n!} z^{(\alpha-n)/2}J_{\alpha-n}(2\sqrt{z}). \quad (36)$$

Examples 8b: (10; special cases of formula (37): $\alpha = 0$, $\beta = 0$, de Louville, Legendre; $\alpha = \beta$, Gegenbauer; known special case of formula (38): $m = 0$, $t = 1/x$; special case of formula (39): $\alpha = 0$, $\beta = 0$, Didon)

$$(t^2-2tx+1)^{-(\alpha+1)/2}P_\alpha^\beta\left(\frac{x-t}{\sqrt{t^2-2tx+1}}\right) = \sum_{n=0}^{\infty}\binom{\alpha-\beta+n}{n}t^nP_{\alpha+n}^\beta(x). \quad (37)$$

$$(t^2-2tx+1)^{p/2}P_p^m\left(\frac{x-t}{\sqrt{t^2-2tx+1}}\right) = \sum_{r=0}^{p-m}\binom{p+m}{r}(-)^rt^rP_{p-r}^m(x). \quad (38)$$

$$(t^2-2tx+1)^{-(\alpha+1)/2}Q_\alpha^\beta\left(\frac{x-t}{\sqrt{t^2-2tx+1}}\right) = \sum_{n=0}^{\infty}\binom{\alpha-\beta+n}{n}t^nQ_{\alpha+n}^\beta(x) \quad (39)$$

Example 8c: (5; Sonine, Pinney)

$$(1-t)^{-\alpha-\gamma-1}\exp\left(-\frac{xt}{1-t}\right)L_\alpha^{(\gamma)}\left(\frac{x}{1-t}\right) = \sum_{n=0}^{\infty}\binom{\alpha+n}{n}t^n L_{\alpha+n}^{(\gamma)}(x). \quad (40)$$

THEOREM 9. *If $F(z, \alpha)$ is a solution of the F-equation, then*

$$\sum_{n=0}^{\infty} F(z, \alpha + n)t^n = \int_0^{\infty} e^{-\theta} F(z + \theta t, \alpha)d\theta, \tag{41}$$

provided that both the series and the integral converge uniformly.

Example 9a: (8)

$$\frac{\sqrt{\pi}}{2t} \exp\left[\left(\frac{1}{2t} - z\right)^2\right] \operatorname{erfc}\left(\frac{1}{2t} - z\right) = \sum_{n=0}^{\infty} t^n H_n(z). \tag{42}$$

Example 9b: (1, and a result of Schlömilch)

$$\int_0^{\infty} e^{-\theta} \left\{ J_0\sqrt{x^2 - 2\theta\tau x} + J_0\sqrt{x^2 + \frac{2\theta x}{\tau}} \right\} d\theta =$$

$$J_0(x) + \exp\left[x\left(\tau - \frac{1}{\tau}\right)/2\right]. \tag{43}$$

THEOREM 10. *If $F(z, \alpha)$ is a solution of the F-equation such that, for fixed values of α and z,*

$$\overline{\operatorname{Lim}}_{n\to\infty} \frac{F(z, \alpha - n - 1)}{F(z, \alpha - n)} = \frac{1}{k}, \qquad k = k(z, \alpha), \tag{44}$$

and if

A. $F(z_1, \alpha) = 0$, $\alpha = \alpha_0 + i$, $i = 0, 1, 2, \ldots$, and $F(z, \alpha)$ is integrable over the range $(0, z_1 - z)$, then when $|t| < k$

$$\sum_{n=0}^{\infty} F(z, \alpha - n)t^n = \int_{z_1}^{z} e^{(z-\xi)t} F(\xi, \alpha + 1)d\xi; \tag{45}$$

B. $\int_0^{\infty} F(z + \theta, \alpha)d\theta$ exists, but $F(z, \alpha+1)$ does not vanish, then when $|t| < k$

$$\sum_{n=0}^{\infty} F(z, \alpha - n)t^n = -\int_0^{\infty} e^{-\theta t} F(z + \theta, \alpha + 1)d\theta; \tag{46}$$

C. (Appell) *$F(z, \alpha + 1) \equiv 0$, then when $|t| < k$*

$$\sum_{n=0}^{\infty} F(z, \alpha - n)t^n = e^{zt} \sum_{n=0}^{\infty} F(0, \alpha - n)t^n. \tag{47}$$

Examples 10a: (4; formula (50): Deruyts)

$$\sum_{n=0}^{\infty} \frac{z^{-\alpha+n} L_\gamma^{(-\alpha+n)}(z)}{\Gamma(-\alpha + n + \gamma + 1)} t^n = \frac{1}{\Gamma(\gamma - \alpha)} \int_0^z e^{(z-\xi)t} \cdot$$

$$\xi^{-\alpha-1} L_\gamma^{(-\alpha-1)}(\xi)d\xi, \quad \Re\alpha < 0. \tag{48}$$

$$\sum_{n=0}^{\infty} \frac{z^{-\alpha+n} L_p^{(-\alpha+n)}(z)}{\Gamma(-\alpha + n + p + 1)} t^n = \frac{1}{\Gamma(p - \alpha)}.$$

$$\int_0^\infty e^{-\theta t}(z+\theta)^{-\alpha-1}L_p^{(-\alpha-1)}(z+\theta)d\theta,\ \Re\alpha > p+1,\ \alpha \text{ not an integer}; \quad (49)$$

$$\sum_{n=0}^{\infty} \frac{z^{-p+n}L_p^{(-p+n)}(z)}{n!}\, t^n = \frac{(t-1)^p}{p!}\, e^{zt}. \quad (50)$$

Examples 10b: (6; Sonine)

$$\sum_{n=0}^{\infty} L_n^{(\gamma)}(x)\, \frac{t^n}{\Gamma(\gamma+n+1)} = e^t(xt)^{-\alpha/2}J_\alpha(2\sqrt{xt}). \quad (51)$$

Example 10c: (11; case $m = 0$: Catalan)

$$\sum_{n=0}^{\infty} \frac{t^n}{(m+n)!}\, P_{m+n}^m(x) = e^{xt}(1-x^2)^{m/2}\, \frac{2^m[\Gamma(m+1/2)]^2}{\pi m!}\cdot$$

$$\cdot {}_2F_3\left(m+\tfrac{1}{2}, m+\tfrac{1}{2};\ \tfrac{1}{2}, m/2+\tfrac{1}{2}, m/2+1;\ -\frac{t^2}{4}(1-x^2)\right). \quad (52)$$

THEOREM 11. *Suppose that $F(z, \alpha)$ is a solution of the F-equation such that*

(i) $\int_0^c t^{\alpha+k}F(t, \alpha)\, dt$ *exists and represents an analytic function of* α *in some strip* $\alpha_0 \leqq \Re\alpha \leqq \alpha_0 + 1$, $\alpha_0 + \Re k > -1$;

(ii) $F(t, \alpha)$ *is a continuous function of* t *when* $0 \leqq t \leqq c$, $\alpha_0 \leqq \Re\alpha \leqq +1$;

(iii) $F(c, \alpha) = 0, \qquad \alpha_0 \leqq \Re\alpha \leqq \alpha_0 + 1$;

(iv) $e^{-i\alpha\pi}\int_0^c t^{\alpha+k+1}F(t, \alpha)dt/\Gamma(\alpha+k+1)$ *is a bounded function of* α *everywhere in the strip* $\alpha_0 \leqq \Re\alpha \leqq \alpha_0 + 1$.

Then

$$\int_0^c t^{\alpha+k}F(t, \alpha)dt = e^{i\alpha\pi}\Gamma(\alpha+k+1)f(k),\ (\Re\alpha + k) > -1. \quad (53)$$

Example 11a: (1; Lipschitz, Weber)

$$\int_0^\infty t^{\alpha/2+k}J_\alpha(2\sqrt{t})dt = \frac{\Gamma(\alpha+k+1)}{\Gamma(-k)},\ \Re(-k-1/2) > \Re\alpha > -1. \quad (54)$$

Example 11b: (8)

$$\int_0^\infty e^{-t^2}t^\alpha H^k{}_\alpha(-t)dt = \frac{e^{i\alpha\pi}\Gamma(\alpha+k+1)\Gamma(1/2)}{2^{k+1}\Gamma(k/2+1)},$$

$$\Re(\alpha+k) > 1. \quad (55)$$

THEOREM 12. *Let $F(z, \alpha)$ be a solution of the F-equation such that $F(z_0, \alpha) = \phi(\alpha)$. Then when the integral and the series are both uniformly convergent,*

$$\int_0^\infty e^{-(At)^{1/m}}t^{\alpha+\gamma-1}F((z-z_0)t+z_0, \alpha)dt =$$

$$\frac{m}{A^{\alpha+\gamma}}\sum_{n=0}^{\infty}\frac{\Gamma(m\alpha+m\gamma+mn)}{n!}\,\phi(\alpha+n)\left(\frac{z-z_0}{A}\right)^n. \quad (56)$$

Example 12a: (1; cases $m = 1, 2$: Lipschitz, Weber, Hankel, Hobson, and others)

$$\int_0^\infty e^{-(At)^{1/m}} t^{\alpha/2+\gamma-1} J_\alpha(2\sqrt{zt})dt = \frac{mz^{\alpha/2}\Gamma(m\gamma+m\alpha)}{A^{\alpha+\gamma}\Gamma(\alpha+1)} {}_mF_1\left(\alpha+\gamma, \alpha+\gamma+\frac{1}{m}, \ldots, \alpha+\gamma+\frac{m-1}{m}; \alpha+1; -\frac{zm^m}{A}\right). \quad (57)$$

Example 12b: (3; case $m = 1, \delta = 1, z = 1$: Sonine)

$$\int_0^\infty e^{-z-(At)^{1/m}} t^{\alpha+\delta-1} L_\gamma^{(\alpha)}(zt)dt = \frac{m\Gamma(m\alpha+m\delta)\Gamma(\alpha+\gamma+1)}{A^{\alpha+\delta}\Gamma(\alpha+1)\Gamma(\gamma+1)} {}_{m+1}F_1\left(\alpha+\delta, \alpha+\delta+\frac{1}{m}, \ldots, \alpha+\delta+\frac{m-1}{m}, \alpha+\gamma+1; \alpha+1; -\frac{zm^m}{A}\right). \quad (58)$$

THEOREM 13. *If $F(z, \alpha)$ is a solution of the F-equation, then when both the series and the integral are uniformly convergent,*

$$\int_0^1 t^{\alpha+\beta-1}(1-t^{1/m})^{m\gamma-1} F((z-z_0)t+z_0, \alpha)dt = m\Gamma(m\gamma)\sum_{n=0}^{\infty} \frac{\Gamma(m\alpha+m\beta+mn)}{\Gamma(m\alpha+m\beta+m\gamma+mn)} \phi(\alpha+n)\frac{(z-z_0)^n}{n!}. \quad (59)$$

Example 13a: (1; case $m = 1, \beta = 1$: Sonine)

$$\int_0^1 t^{\alpha/2+\beta-1}(1-t^{1/m})^{m\gamma-1} J_\alpha(2\sqrt{zt})dt = \frac{mz^{\alpha/2}\Gamma(m\alpha+m\beta)\Gamma(m\gamma)}{\Gamma(m\alpha+m\beta+m\gamma)\Gamma(\alpha+1)} {}_mF_{m+1}\left(\alpha+\beta, \alpha+\beta+\frac{1}{m}, \ldots, \alpha+\beta+\frac{m-1}{m}; \alpha+\beta+\gamma, \alpha+\beta+\gamma+\frac{1}{m}, \ldots, \alpha+\beta+\gamma+\frac{m-1}{m}, \alpha+1; -z\right). \quad (60)$$

Example 13b: (3; case $\delta = \alpha + 1$: Koshliakoff)

$$\int_0^1 t^{\beta-1}(1-t)^{\gamma-1} L_\epsilon^{(\alpha)}(zt)dt = \frac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\alpha+\epsilon+1)}{\Gamma(\alpha+1)\Gamma(\delta+1)\Gamma(\delta+\gamma)} {}_2F_2(-\epsilon, \delta; \alpha+1, \delta+\gamma; z). \quad (61)$$

THEOREM 14. *Suppose $F(z, \alpha)$ is a solution of the F-equation, and suppose the functions $F_i(z, \alpha)$ form a set of solutions of the F-equation. Let $\mathbf{O}_t$ be an operator which binds the variable t and which commutes, in the formula (63) below, with the operations of differentiation with respect to z, replacing α by $\alpha + 1$, and replacing z by z_0. Then if the relation*

$$F(z_0, \alpha) = \mathbf{O}_t[F_t(z_0, \alpha)], \qquad \Re\alpha \geqq \alpha_0, \quad (62)$$

holds for a fixed value z_0 of z, it holds for all values of z such that it has a meaning:

$$F(z, \alpha) = \mathbf{O}_t[F_t(z, \alpha)], \qquad \Re\alpha \geqq \alpha_0. \tag{63}$$

COROLLARY 14A. *Let $F_1(z, \alpha)$ and $F_2(z, \alpha)$ be two solutions of the F-equation. Suppose that there exists an operator $\mathbf{O}_t$ as in Theorem 14 and a function $G(t)$ such that*

$$F_2(z_0, \alpha) = F_1(z_0, \alpha)\mathbf{O}_t[\{G(t)\}^\alpha], \qquad \Re\alpha \geqq \alpha_0. \tag{64}$$

Then if $\mathbf{O}_t$ has the further property $\mathbf{O}_t[f(z, \alpha)g(t)] = f(z, \alpha)\mathbf{O}_t[g(t)]$, it follows that

$$F_2(z, \alpha) = \mathbf{O}_t[\{G(t)\}^\alpha F_1((z - z_0)G(t) + z_0, \alpha)], \qquad \Re\alpha \geqq \alpha_0, \tag{65}$$

for all values of z for which the expression on the right has a meaning.

Example 14Aa: Problem: to find a formula expressing $L_\gamma^{(\alpha)}(x)$ in terms of $J_\alpha(x)$. Consider solutions nos. 1 and 3. Substitute in formula (64) when $z = 0$. Then we need to find $G(t)$ and $\mathbf{O}_t$ such that

$$\Gamma(\alpha + \gamma + 1)/\Gamma(\gamma + 1) = \mathbf{O}_t[\{G(t)\}^\alpha].$$

A suitable selection is given by the formulae

$$G(t) \equiv t,$$

$$\mathbf{O}_t[\ldots] \equiv \frac{1}{\Gamma(\gamma + 1)} \int_0^\infty e^{-t}t^\gamma[\ldots]dt.$$

Formula (65) then gives Hankel's integral

$$e^{-z}z^{\alpha/2}L_\gamma^{(\alpha)}(z) = \frac{1}{\Gamma(\gamma + 1)} \int_0^\infty e^{-t}t^{\gamma+\alpha/2} J_\alpha(2\sqrt{zt})dt. \tag{66}$$

To invert this integral, interchange the rôles of solutions 1 and 3 in formula (64). We then find a generalization of formula (30):

$$z^{-\alpha/2} J_\alpha(2\sqrt{z}) = \frac{\Gamma(\gamma + 1)}{2\pi i} \int_{-\infty}^{(0+)} e^{t-z/t}t^{-\alpha-\gamma-1} L_\gamma^{(\alpha)}\left(\frac{z}{t}\right)dt. \tag{67}$$

Example 14Ab: Problem: to find a formula expressing $J_{\alpha+\beta}(x)$ in terms of $J_\alpha(x)$. We use the solution no. 1 for both $F_1(z, \alpha)$ and $F_2(z, \alpha)$ in Corollary 14A. Using the definitions

$$G(t) \equiv \sin^2 t$$

$$\mathbf{O}_t[\ldots] \equiv \frac{e^{i\beta\pi}}{2\Gamma(\beta)} \int_0^{\pi/2} \cos^{2\beta-1}t \sin t[\ldots]dt,$$

we discover Sonine's formula

$$\left(\frac{x}{2}\right)^{-\beta} J_{\alpha+\beta}(x) = \frac{1}{2\Gamma(\beta)} \int_0^{\pi/2} J_\alpha (x \sin t) \sin^{2\alpha+1} t \cos^{2\beta-1} t \, dt. \quad (68)$$

Example 14Ac: Problem: to find a formula giving the Hermite functions in terms of the Legendre functions. We use solutions nos. 3 and 10 and the choices

$$G(t) \equiv \sqrt{t},$$
$$\mathbf{O}_t[\ldots] \equiv \frac{2^\beta}{\sqrt{\pi}} \int_0^\infty e^{-t} t^{-1/2+\beta/2} [\ldots] dt.$$

We find then that

$$(1 - x^2)^{\beta/2} P_\alpha^\beta(x) = \frac{2^{\beta+1}}{\sqrt{\pi}\Gamma(\alpha - \beta + 1)} \int_0^\infty e^{-\theta^2} \theta^{\alpha+\beta} H_{\alpha-\beta}(x\theta) d\theta. \quad (69)$$

Similar methods enable us to find an inverse:

$$H_\gamma(z) = \frac{\Gamma(\gamma + 1)}{2\beta\sqrt{\pi i}} \int_{+\infty}^{-\infty} e^{-\theta^2} (-\theta^2 - z^2)^{-\beta/2} (-\theta)^{-\beta-\gamma} P_{\beta+\gamma}^\beta \left(\frac{iz}{\theta}\right) d\theta. \quad (70)$$

Case $\beta = 0$: Glaisher, Curzon.

COROLLARY 14B. *If $F_1(z,\alpha)$ and $F_2(z, \alpha)$ are solutions of the F-equation, and if for some value of z the relation*

$$F_2(z, \alpha) = \mathbf{O}_t[F_1(z + t, \alpha)], \qquad \Re\alpha \geqq \alpha_0, \quad (71)$$

is correct, then it is correct for all values of z for which the expression on the right has a meaning.

Example 14B: (4, and a special case of formula (58); case $z = 0$: Sonine)

$$\int_0^\infty e^{-yt} (t + z)^\alpha L_\gamma^{(\alpha)} (t + z) dt = \frac{\Gamma(\alpha + \gamma + 1)(y - 1)^\gamma}{\Gamma(\gamma + 1) y^{\gamma+\alpha+1}} e^{zy}. \quad (72)$$

There are other interesting special cases of Theorem 14 not included in these two corollaries.

THEOREM 15 (Doetsch). *Suppose $F(z, \alpha)$ is any solution of the F-equation such that $F(z_0, \alpha_0 - j), j = 0, 1, 2, \ldots, m$, exists. Then*

$$F(z, \alpha_0 - m) = \int_{z_0}^z F(t, \alpha_0) \frac{(z - t)^{m-1}}{(m - 1)!} dt + \sum_{j=1}^m F(z_0, \alpha_0 - j) \frac{(z - z_0)^{m-j}}{(m - j)!} \quad (73)$$

Examples 15a: (1, 2; Sonine)

$$z^{-(\alpha-m)/2} J_{\alpha-m}(2\sqrt{z}) = \frac{1}{\Gamma(m)} \int_0^\infty t^{m-1}(z+t)^{-\alpha/2} J_\alpha(2\sqrt{z+t})dt. \quad (74)$$

$$z^{(\alpha+m)/2} J_{\alpha+m}(2\sqrt{z}) = \frac{1}{\Gamma(m)} \int_0^z t^{\alpha/2} (z-t)^{m-1} J_\alpha(2\sqrt{t})dt. \quad (75)$$

Example 15b: (5; Koshliakoff)

$$z^{m+\alpha} L_\gamma^{(\alpha+m)}(z) = \frac{\Gamma(\alpha+m+\gamma+1)}{\Gamma(m)} \int_0^z t^\alpha (z-t)^{m-1} L_\gamma^{(\alpha)}(t)dt. \quad (76)$$

Example 15c: (10)

$$(x^2-1)^{\alpha/2} P_{\alpha-m}^\beta(x) = m \binom{\alpha-\beta}{m} \int_{-1}^x (\theta^2-1)^{(\alpha-m-1)/2} (\theta\sqrt{x^2-1} - x\sqrt{\theta^2-1})^{m-1} P_\alpha^\beta(\theta)d\theta. \quad (77)$$